Lecture 13 (Basics of graph theory)

1 Basic Terminologies

Definition 1 A graph G consists of two sets V and E, where V is a non-empty set, called the vertex set, and E is called the edge set. A graph G is also denoted as $G = (V, E)$. We define some terminologies in a graph G as follows.

1. The number $|V|$ is called the order of the graph G and is denoted as $|G|$. By $||G||$, we denote the number of edges in G. A graph with m vertices and n edges is called an (m, n) graph.

2. A graph is said to be non-trivial if it has at least one edge; otherwise, it is called a trivial graph.

3. Let $u, v \in V$. An edge $e \in E$ joining them is denoted as $e = u\,v$. In this case, u and v are called adjacent vertices, also called the end vertices, of the edge e. We say that an edge e is incident on a vertex u if u is an end vertex of the edge e. If two vertices u and v are adjacent, we denote $u \sim v$; otherwise, $u \not\sim v$. Two edges $e_1, e_2 \in E$ are called adjacent if they have a common end vertex.

4. Let $v \in V$. The neighborhood $N(v)$ of v is $N(v) = \{u \in V\ : u \sim v\}$. If $A \subseteq V$, then $N(A)$ is defined as $N(A) = \cup_{v \in A} N(v)$.

5. If the end vertices of an edge $e \in E$ are the same, then the edge is called a loop. If e_1, e_2 are two edges such that they have the same end vertices, then the edges are called parallel edges or multiple edges. A graph is called simple if it has no loops or multiple edges. Unless stated otherwise, by a graph, we mean a simple graph.

6. The degree $\deg(v)$ of a vertex v of a simple graph is defined as $|N(v)|$. A vertex v is called an isolated vertex if $\deg(v) = 0$ and a pendent vertex if $\deg(v) = 1$.

7. G is called k-regular if $\deg(v) = k$ for every vertex. A 3-regular graph is called cubic.

8. A set of vertices or edges is said to be independent if no two of them are adjacent. The maximum size of an independent vertex set is called the independence number of G, denoted $\alpha(G)$.

Example: Consider the above graph G. The vertices 1, 8, and 9 are pendent vertices. The vertex 10 is isolated. Since $N(2) = \{1, 3, 5, 6, 7\}$, $\deg(2) = 5$. Clearly $\delta(G) = 0$ and $\Delta(G) = 5$. Note that $\alpha(G) = 6$, as the set $\{1, 3, 5, 7, 9, 10\}$ is a maximum independent set.

Definition 2 Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$. Then

1. G is called *complete graph*, denoted as K_n, if every pair $\{v_i, v_j\}$, $i \neq j$, forms an edge.

2. G is called *path graph*, denoted as P_n, if $E(G) = \{v_i v_{i+1}\ :\ 1 \leq i \leq n-1\}$.

3. G is called *cycle graph*, denoted as C_n, if $E(G) = \{\{v_i v_{i+1}\ :\ 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$.

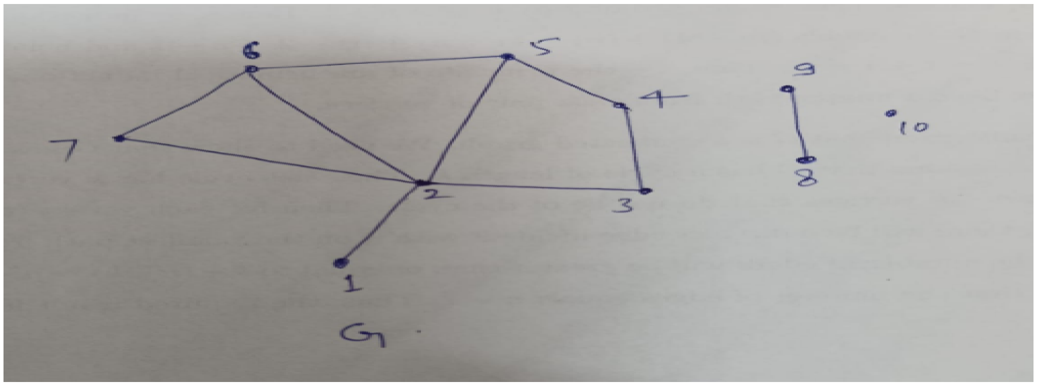

Figure 1:

4. G is called *bipartite graph* if $V = V_1 \cup V_2$ such that $V_1, V_2 \neq \emptyset$, $V_1 \cap V_2 = \emptyset$ and each edge $e \in E$ has one end vertex in V_1 and other in V_2. G is called *complete bipartite graph* if G is a bipartite graph with partite sets V_1 and V_2 such $u \sim v$, $\forall\ u \in V_1$ and $u \in V_2$. If $|V_1| = m$ and $|V_2| = n$, G is denoted by $K_{m,n}$.

Question. What is the maximum number of edges possible in a simple graph of order n?

Lemma 1 [**Handshake Lemma:**] If $G = (V, E)$ is a simple graph, $\sum_{v \in V} \deg(v) = 2|E|$.

Proof: Note that each edge contribute 2 to the sum $\sum_{v \in V} \deg(v)$. Hence, $\sum_{v \in V} \deg(v) = 2|E|$.

Corollary 1 *Let $G = (V, E)$. Then the number of odd-degree vertices is even.*

Proof: Follows from the above lemma.

Question. There are 7 persons in a party. Prove that someone must have an even number of friends, assuming that friendship is mutual.

Solution. Let $P_1, P_2, \cdots, P_7$ are seven persons. Let P_i be a friend of P_j if P_i shakes hand with P_j. Then, by the above corollary,

Sum of the number of shaking hands = 2 × number of couple friends (which is even).

Thus Left hand side can not be even for all.

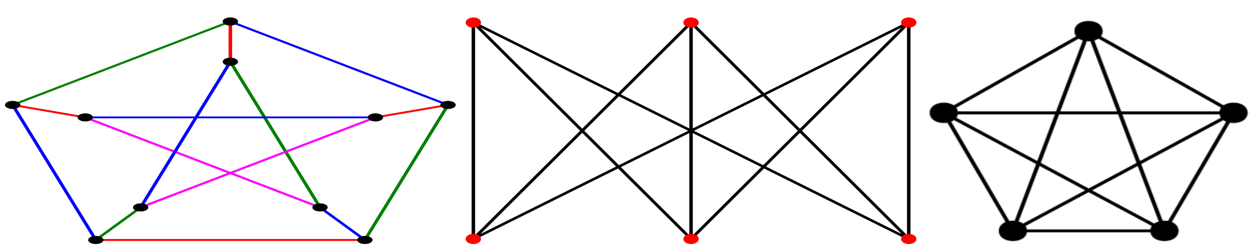

Figure 2: Petersen graph, Complete bipartite graph $K_{3,3}$, Complete graph K_5

Proposition 1 In a graph $G = (V, E)$ with $|V| = n \geq 2$, there are two vertices of equal degree.

Proof: If G has two or more isolated vertices, then we are done. Suppose G has one isolated vertex. Then the remaining $n-1$ vertices have degree between 1 to $(n-2)$ and hence by the Pigeonhole Principle (PHP), the result holds. Otherwise, G has no isolated vertices. Then there are n vertices whose degrees lie between 1 to $n-1$. Again by PHP, the result holds.

Theorem 1 A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof: First assume that $G = (V, E)$ is bipartite graph with bipartite subsets V_1 and V_2 of V. Then assign one color to each vertex of V_1 and a second color to each vertex of V_2 will give the desired condition.

Conversely, let it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color. Let V_1 be the set of vertices assigned one color and V_2 be the set of vertices assigned the other color. Then, V_1 and V_2 are disjoint and $V = V_1 \cup V_2$. □

Definition. Let $G = (V, E)$ be a graph. Then

1. the minimum degree of a vertex in G is denoted by $\delta(G)$, and the maximum degree of a vertex in G is denoted by $\Delta(G)$.
2. G is called k-regular if $\deg(v) = k$ for all $v \in V$.
3. a 3-regular graph is called cubic.
4. The graph P_4 is not regular.

Example.

1. The cycle graph C_n is 2-regular whereas the complete graph K_n is $(n-1)$-regular.
2. The Petersen graph, the complete graph K_4, and the complete bipartite graph $K_{3,3}$ are cubic.

Question. Can we have a cubic graph on 5 vertices?

2 Construction of new graphs from existing graphs

Definition 3 Let $G = (V(G), E(G))$ be a graph. Then

1. A graph $H = (V(H), E(H))$ is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
2. A subgraph $H = (V(H), E(H))$ of G is called a spanning subgraph of G if $V(H) = V(G)$.
3. A subgraph $H = (V(H), E(H))$ of $G = (V, E)$ is called an induced subgraph of G if for every $u, v \in V(H)$, $e = uv \in E(H)$ whenever $e = uv \in E(G)$.
4. If $v \in V$, then the graph $G - v$, called the *vertex deleted subgraph*, is obtained from G by deleting v and all the edges that are incident with v.
5. If $e \in E$, then the graph $G - e = (V, E \setminus \{e\})$ is called the *edge deleted subgraph*.
6. If $u, v \in V$, then $G + uv = (V, E \cup \{uv\})$ is called the graph obtained by edge addition.

7. The complement $\overline{G}$ of a graph G is defined as $(\overline{V}, \overline{E})$, where $\overline{V} = V$ and $\overline{E} = \{uv \,|\, u \neq v, uv \notin E\}$.

Question.

1. In any graph G, $||G|| + ||\bar{G}|| = {}^{|G|} C_2$.
2. In any graph G of order n, $\deg_G(v) + \deg_{\bar{G}}(v) = n - 1$. Thus $\Delta(G) + \Delta(\bar{G}) \geq n - 1$.
3. Characterize graphs G such that $\Delta(G) + \Delta(\bar{G}) = n - 1$.
4. Can we have a graph G such that $\Delta(G) + \Delta(\bar{G}) = n$?
5. Show that a k-regular simple graph on n vertices exists if and only if kn is even and $n \geq k + 1$.

Definition 4 Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs.

1. Then their intersection, denoted $G \cap H$, is defined as $(V(G) \cap V(H), E(G) \cap E(H))$.
2. Then their union, denoted $G \cup H$, is defined as $(V(G) \cup V(H), E(G) \cup E(H))$.
3. Then their Cartesian product, denoted $G \times H$, has $V(G) \times V(H)$ as the vertex set and the edge set consists of all elements $\{uv, u'v'\}$, whether either $u = u'$ and $\{v, v'\} \in E(H)$, or $v = v'$ and $\{u, u'\} \in E(G)$.

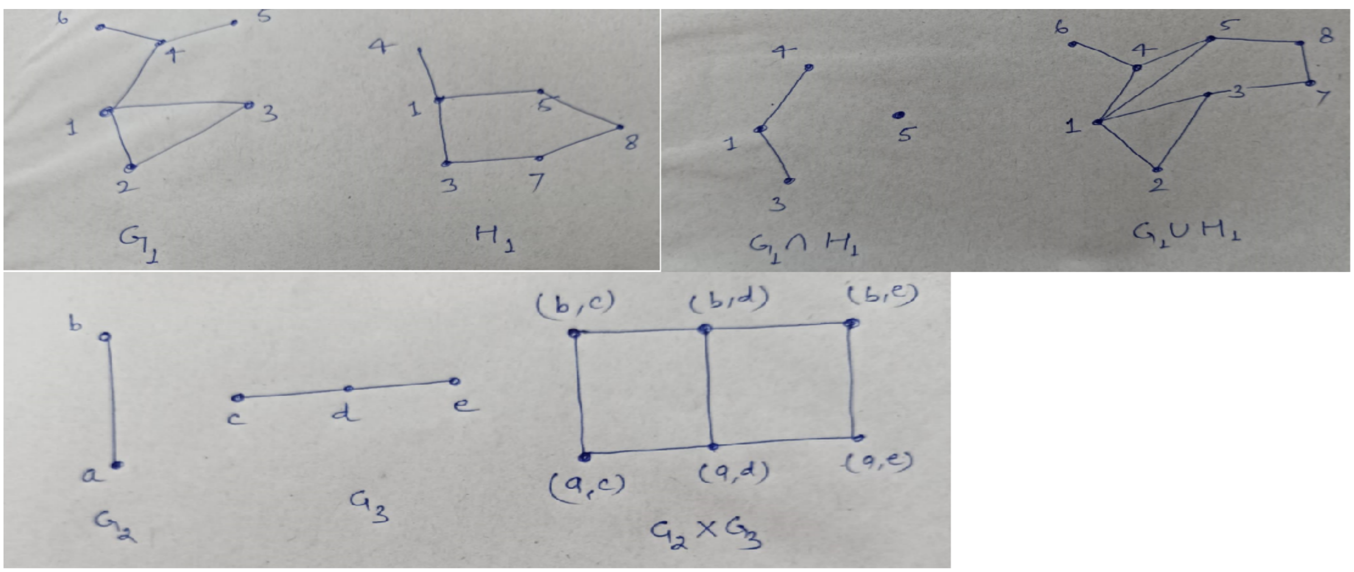

Figure 3: Intersection, Union, and Product of graphs